Ecole Polytechnique
ECO 560 Economics of Contracts

Raicho Bojilov
1 March 2013

Class Notes: Week 8

1 Multitasking

To analyze the problem, we use the simplified setting from the analysis of linear contracts. There are n tasks $i = 1, ..., n$. each associated with an effort level e_i and a noisy signal y_i. The technology that generates the signal is:

$$y_i = e_i + \varepsilon_i$$

The random component is normally distributed with a variance covariance matrix $\sum$ and a (i, j) term σ_{ij}.The compensation is a linear function of the tasks:

$$w = \alpha + \sum_{i=1}^{n} \beta_i y_i$$

Let $n = 2$. The disutility of effort takes the following form:

$$\psi(e_1, e_2) = \frac{1}{2} c_1 e_1^2 + \frac{1}{2} c_2 e_2^2 + \delta e_1 e_2$$

where δ indicates how the effort on one task affects the cost of effort on another. δ is the main source of interdependence between the tasks. We assume that the noise for each task is independent from the others.

The Agent is risk-averse with CARA preferences:

$$u(w, e_1, e_2) = -\exp\{-\eta[w - \psi(e_1, e_2)]\}$$

where $\eta > 0$ is the agent's coefficient of absolute risk-aversion. The Principal is risk-neutral.

1. State the problem of the Principal.

Solution: The Principal's problem is:

$$\max_{e,\alpha,\beta} E(y_1 + y_2 - w)$$

subject to

$$E(-\exp\{-\eta[w - \psi(e_1, e_2)]\}) = u(\underline{w})$$

and

$$(e_1, e_2) \in \arg\max\{E(-\exp\{-\eta[w - \psi(e_1, e_2)]\})\}$$

2. Use the fact that if $\varepsilon_i \sim N(0, \sigma_i^2)$,

$$E(\exp\{\gamma\varepsilon_i\}) = \exp\left\{\frac{1}{2}\gamma^2\sigma_i^2\right\}$$

which implies that

$$E(\exp\{\eta\beta\varepsilon_i\}) = \exp\left\{\frac{1}{2}\eta^2\beta^2\sigma_i^2\right\}$$

to restate the restate the problem of the Agent in terms of the certainty equivalent.

Solution: Then maximizing the Agent's utility is then equivalent to maximizing the certainty equivalent defined by:

$$\begin{aligned} & -\exp\left\{-\eta\left[\alpha + \beta_1 e_1 + \beta_2 e_2 - \psi(e_1, e_2) - \frac{1}{2}\eta\left(\beta_1^2\sigma_1^2 + \beta_2^2\sigma_2^2\right)\right]\right\} \\ = & -\exp\{-\eta\overline{w}(e)\} \end{aligned}$$

because the exponent function is monotone.

The modified problem of the Agent is:

$$\max_{e_1,e_2}\left[\alpha + \beta_1 e_1 + \beta_2 e_2 - \frac{1}{2}c_1 e_1^2 - \frac{1}{2}c_2 e_2^2 - \delta e_1 e_2 - \frac{1}{2}\eta\left(\beta_1^2\sigma_1^2 + \beta_2^2\sigma_2^2\right)\right]$$

3. Solve for the optimal effort for each task in the Agent's problem, given a compensation schedule. Assume that the solution is interior, i.e. $e_i > 0$ for $i = 1, 2$.

Solution: The first-order conditions for this problem yield:

$$e_i = \frac{\beta_i c_j - \delta \beta_j}{c_i c_j - \delta^2}$$

4. Use the solution of the Agent's problem to eliminate effort from the problem of the Principal and the base pay α.

Solution: Substituting this expression for effort into the Principal's problem, we obtain:

$$\max_{\alpha, \beta_1, \beta_2} \left\{ \sum_{i=1}^{2} \left(\frac{\beta_i c_j - \delta s_j}{c_i c_j - \delta^2} \right) (1 - \beta_i) - \alpha \right\}$$

subject to

$$\left[\alpha + \beta_1 e_1 + \beta_2 e_2 - \psi(e_1, e_2) - \frac{1}{2} \eta \left(\beta_1^2 \sigma_1^2 + \beta_2^2 \sigma_2^2 \right) \right] = \underline{w}$$

Using the IR constraint, we can express base pay in terms of the other variables:

$$\alpha = \underline{w} - \left(\beta_1 e_1 + \beta_2 e_2 - \psi(e_1, e_2) - \frac{1}{2} \eta \left(\beta_1^2 \sigma_1^2 + \beta_2^2 \sigma_2^2 \right) \right)$$

The problem of the Principal can be solved using unconstrained optimization methods:

$$\max_{\beta_1, \beta_2} \sum_{i=1}^{2} \left(\frac{\beta_i c_j - \delta s_j}{c_i c_j - \delta^2} \right) (1 - \beta_i)$$
$$- \left(\underline{w} - \left(\beta_1 e_1 + \beta_2 e_2 - \psi(e_1, e_2) - \frac{1}{2} \eta \left(\beta_1^2 \sigma_1^2 + \beta_2^2 \sigma_2^2 \right) \right) \right)$$

5. Find the first order conditions associated with the reformulated problem and express the bonus rates in terms of the model parameters.

Solution: The first-order condition for this problem yields the following system of two equations:

$$\beta_i = \frac{c_j - \delta + \delta \beta_j}{c_j + \eta \sigma_i^2 \left(c_i c_j - \delta^2 \right)}$$

Solving the two equations, we obtains:

$$\beta_1 = \frac{1 + (c_2 - \delta)\,\eta\sigma_2^2}{1 + \eta c_2\sigma_2^2 + \eta c_1\sigma_1^2 + \eta^2\sigma_1^2\sigma_2^2\left(c_1c_2 - \delta^2\right)}$$

$$\beta_2 = \frac{1 + (c_1 - \delta)\,\eta\sigma_1^2}{1 + \eta c_2\sigma_2^2 + \eta c_1\sigma_1^2 + \eta^2\sigma_1^2\sigma_2^2\left(c_1c_2 - \delta^2\right)}$$

6. Consider the problem when $\delta = 0$ and simplify the incentive structure accordingly. Interpret.

Solution: Note that when $\delta = 0$, the optimal effort choice for task i is:

$$e_i = \frac{\beta_i}{c_i}$$

so the optimal slopes of the incentives are reduced to:

$$\beta_i = \frac{1}{1 + \eta c_i\sigma_i^2}$$

7. Discuss how the incentives depend on the noisiness of the signals, risk-aversion, and disutility of effort.

Comments on the solution:

- The base pay α depends on the reservation wage $\underline{w}$ and β.
- The bonus rate for a task depends on the properties of the other task.
- As σ_1^2 increases, the bonus rate β_1 decreases: the signal is less informative, so compensation depends less on it.
- As σ_1^2 increases, the bonus rate for task 2 also decreases! You can verify this by checking $\frac{\partial\beta_2}{\partial\sigma_1^2} < 0$. This is driven by the complementarity of β_1 and β_2 when there is substitution between efforts e_1 and e_2.
- As $\sigma_1^2 \to \infty$, compensation becomes a flat wage. Think of $e_i = 0$ as some form of normalization rather than a true corner solution i.e. it is the effort level that people would exert when there are no incentives. Then, the optimal contract is to set the bonuses to 0. Any incentives based on the observable output with finite variance, will lead to a shift of effort away from the task with infinite variance, which results in a lower profits. This is a potential explanation of why in complex environments, we observe fixed compensation.

2 Tournaments

Promotions and rankings of employees are very common in the real world. Yet, if the noisy outcomes signals for two agents are independent from each other, one may expect that such "tournament" compensation schemes should not be optimal. The following example, however, illustrates that this may not be the case. Actually, they may help in achieving socially efficient outcomes. This point is illustrated in the following example.

Consider an environment with unverifiable or unobservable actions. The noisy outcome signal is generated by a technology:

$$y = e + \varepsilon$$

where ε is noise and e is the effort chosen by the Agent. Suppose that both the Principal and the Agent are risk-neutral. In particular, suppose that the Agent's state utility is given by:

$$u(w, e) = w - \psi(e)$$

where ψ is increasing and convex function.

1. Find the first-best level of effort e.

Solution: From the first-order condition of the social planner for effort:

$$1 = \psi'(e)$$

Let the socially optimal level of effort be denoted e^*, which is the solution of the problem above.

2. Suppose that the Principal implements a linear contract in output y :

$$w = \alpha + \beta y$$

Find the β_i that implements the socially optimal level of effort in an environment of unobservable or unverifiable actions.

Solution: Any contracted action must be optimal for the Agent to take, so from the first order conditions for the utility maximization problem of the Agent, we have:

$$\beta = \psi'(e)$$

We are asked to find the bonus rate that implements the socially optimal level of effort e^*. Combining the conditions from 1 and 2, we have that $\beta = 1$.

3. Find the base pay α under the socially optimal linear contract.

Solution: From the individual rationality constraint of the Agent, we have:

$$\underline{u} = \alpha + e^* - \psi(e^*)$$

which implies that

$$\alpha = \underline{u} - (e^* - \psi(e^*))$$

4. Interpret the results in 2 and 3. Is there a linear contract that implements the socially optimal level of effort? Who captures the surplus from the contract? The Principal or the Agent? What is the mechanism through which this outcome is achieved: the base pay or the bonus rate?

Solution: The Principal captures all the surplus while ensuring that the Agent participates. This outcome is achieved through the base pay, despite that the Agent gets a bonus rate that allows for the capture of all of the marginal product of effort. The outcome is socially efficient.

5. In such an environment would you be for or against minimum wage? What will be the effect of the minimum wage?

Solution: Depends on the utility weights that you put on the welfare of the Principal and of the Agent.

Next, we consider an alternative compensation scheme according to which the Agent gets a base pay α and a reward R if she has the highest output among all employees. Let us assume for simplicity that there are two employees: 1 and 2

6. What is the expected compensation of employee 1?

Solution:

$$\begin{aligned} E(w) &= \alpha + \Pr(y_1 \geq y_2).R \\ &= \alpha + \Pr(e_1 - e_2 \geq \varepsilon_2 - \varepsilon_1) \end{aligned}$$

7. Let the CDF of $\varepsilon_2 - \varepsilon_1$ be F with a pdf f. What is the optimal action for the Agent 1 under this new compensation scheme?

Solution: From the first-order conditions, we have that the optimal effort for the agent is the solution of:

$$R.f(e_1 - e_2) = \psi'(e_1)$$

8. The optimal effort level depends on the effort level of the other workers. Suppose that we are looking for a symmetric Nash equilibrium, i.e. all agents

choose the same level of effort in equilibrium. Find the socially optimal reward R.

Solution: The socially optimal reward implements the socially optimal level of effort e^*. Under symmetric equilibrium $e_1 = e_2$, so the first-order condition for agent 1 becomes:

$$Rf(0) = \psi'(e)$$

At the same time, we know that the choice of effort must be socially optimal so from 1, we simplify the condition to:

$$Rf(0) = 1$$

which implies that the reward is:

$$R = \frac{1}{f(0)}$$

9. Find the profit maximizing base pay.

Solution: The profit maximizing base pay ensures that the Agent gets only the IR level of utility, so

$$\alpha = \underline{u} - \left(\frac{1}{f(0)} F(0) - \psi(e^*) \right)$$

10. Discuss the properties of the socially optimal tournament contract. In particular, who gains most from the surplus associated with the contract? What determines whether someone wins the reward R : effort or luck?

Solution: The Principal captures most of the surplus and promotions are determined entirely by luck, but this scheme is ex ante maximizing incentives.

11. If you had two design the compensation promotions scale in a firm, would you have higher rewards for higher levels in system or decreasing rewards? Discuss the intuition.

Solution: Rewards should increase because the option value associated with winning future rewards declines as people get promotions. This is a possible explanation for CEO compensation and increasing compensation with rank and age.

3 Teams

There are n agents that produce output

$$Y = F(e_1, .., e_n)$$

which is a deterministic function of the effort of each agent. Suppose that F is strictly increasing and concave in $(e_1, ..., e_n)$. Also suppose that efforts are complementary in the sense that:

$$\frac{\partial^2 Y}{\partial e_i \partial e_j} \geq 0$$

Finally, suppose that the agents are risk neutral with a utility function:

$$u(w, e_i) = w - \psi(e_i)$$

The Principal does not observe the output associated with the effort of each individual: only aggregate output is observed. Agents make their effort choice simultaneously and do not coordinate actions.

Any compensation must also satisfy for each Y

$$\sum_i w_i(Y) = Y$$

1. Find the socially optimal effort level for each worker. Find the individually optimal level of effort at the Nash equilibrium of the game that the team members play.

Solution: The optimal effort choice for each agent i in a world of observable actions is given by:

$$\frac{\partial Y}{\partial e_i} = \psi'(e_i)$$

However, in a world of unverifiable or unobservable effort choice, agents choose their effort as a best response to what other agents' effort is:

$$\frac{\partial w_i(Y(e_i, e_{-i}))}{\partial Y} \frac{\partial Y(e_i, e_{-i})}{\partial e_i} = \psi'(e_i)$$

2. Design a mechansim/environment in which the Nash equilibrium of the game coincides with the socially optimal level of effort of the workers.

Solution: Still, it seems possible to obtain the first-best by setting:

$$\frac{\partial w_i\left(Y\left(e_i, e_{-i}\right)\right)}{\partial Y} = 1$$

However, this equation implies that:

$$w_i\left(Y\right) = Y$$

(up to a constant). The key problem then is how to ensure optimal effort, as suggested above and at the same time to satisfy the budget constraint:

$$\sum_i w_i\left(Y\right) = Y$$

One possible solution, suggested by Holmstrom is that of a budget breaker. A third party signs a contract with the agents agreeing to pay them

$$w_i\left(Y\right) = Y$$

At the same time, the third party asks the agents to make an initial contribution in the amount of $\left(n-1\right)Y^*$, where Y^* is the output under first-best effort by all.

This model can be used to explain why firms may need a residual claimant or why firms seek outside financing as a way to break the budget constraint.

A Mirrlees Contract

Let us allow for some noise in aggregate outcomes. Then, the third party loses money when actual outcome is better than expected, which seems odd. A possible alternative solution is to introduce a Mirrlees contract with bonuses β_i and penalties k_i in case any other outcome is realized, such that

$$\beta_i \geq \psi\left(e_i\right) - k_i$$

and

$$Y^* \geq \sum_i \beta_i$$

Another possible interpretation is debt financing by the firm. The firm agrees to pay:

$$D = Y^* - \sum_i \beta_i$$

In case of a default, the creditors collect the residual compensation nk.