Ecole Polytechnique
ECO 560 Economics of Contracts

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Week 4: Class Notes

# 1 Revelation Principle and Bayes-Nash Equilibrium

Suppose that there exists a mechanism $\Gamma = \left((S_i)_{i\in N}, g(.)\right)$ that implements the social choice function $f(.)$ in Bayes-Nash equilibrium. Then $f(\cdot)$ is truthfully implementable in Bayes-Nash equilibrium.

**Solution:** Suppose $\Gamma = \left((S_i)_{i\in N}, g(.)\right)$ implements $f(\cdot)$ in Nash equilibrium. Then there exists s*$(\cdot)$ = (s*$_1(\cdot)$, ..., s*$_N$ $(\cdot)$) such that for all type profiles $\theta$

$$g(s^*(\theta)) = f(\theta) \tag{1}$$

and for all $i$, for all $\theta_i \in \Theta_i$ :

$$u_i(g\left(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})\right), \theta_i) \geq u_i(g\left(s_i', s_{-i}^*(\theta_{-i})\right), \theta_i) \tag{2}$$

for all $s_i'$.

Condition (2) implies in particular that:

$$u_i\left(g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})), \theta_i\right) \geq u_i g\left(s_i\left(\theta_i'\right), s_{-i}^*(\theta_{-i})\right), \theta_i)$$

for all $\theta_i'$. By condition (1), we can rewrite these inequalities as:

$$u_i\left(f(\theta_i, \theta_{-i}), \theta_i)\right) \geq u_i\left(f(\theta_i', \theta_{-i}), \theta_i\right)$$

But this last condition implies that the social choice function can be implemented through a direct truthful mechanism.

## 2 A Simple Signalling Game

Consider the following signalling game. Player 2 can be type I or type II and she knows her type. They payoff matrices associated with this game are:

Payoffs if player 2 is type I:

| 1\2 | C | N |
|---|---|---|
| C | 0,0 | 7,-2 |
| N | -2,7 | 5,5 |

Payoffs if player 2 is type II:

| 1\2 | C | N |
|---|---|---|
| C | -2,-2 | 5,0 |
| N | 0,5 | 7,7 |

The informed player can signal her type to the uninformed player by picking her action before player 1 picks his and having this choice observed by the uninformed player. Suppose that the common prior for type I is $p$>0.5.

**i.** Find all separating PBE.

**Solution:** There are two possible separating equilibria:

| | Case 1 | Case 2 |
|---|---|---|
| Type I | C | N |
| Type II | N | C |

**Case 1:** By definition of separating equilibrium,

$$\mu(I|C) = 1 \text{ and } \mu(II|N) = 1$$

so player 1 knows exactly the correct payoff matrix and the relevant payoff column, C for type I and N for type II. The best response of 1 are $BR_1(C) = C$ and $BR_1(N) = N$. What remains is to check whether type I or type II has an incentive to deviate. If type I does not deviate, then her payoff from C is 0. If she deviates by playing N player 1 will mistake her to be type II and will play N, so type I will get a payoff of 5. Consequently, type I has an incentive to deviate. Thus, there is no separating PBE in which the types send signals (C,N).

**Case 2:** By definition of separating equilibrium,

$$\mu(I|N) = 1 \text{ and } \mu(II|C) = 1$$

Also $BR_1(N) = C$ and $BR_1(C) = N$. Clearly type I has an incentive to deviate to N, so there is no separating equilibrium in which the types send signals (N,C).

**ii.** Find all the pooling pure strategy equilibria.

**Solution:** There are two possible pooling equilibria:

| | Case 1 | Case 2 |
|---|---|---|
| Type I | C | N |
| Type II | C | N |

**Case 1:** By definition of pooling equilibrium, the posterior belief is the same as the prior belief on the equilibrium path: $\mu(I|C) = p$: If the received signals is not C, but N, then $\mu(I|N) = p'$. Consider first the best response of player 1.

Suppose that player 1 observes C, then

$$\pi_1(C) = -2(1-p)$$

$$\pi_1(N) = -2p$$

Since $p > 0.5$, 1 will play C. Now, suppose that player 1 observes N (which may happen if one of the types deviates). Then

$$\pi_1(C) = 7p' + 5(1-p')$$

$$\pi(N) = 5p' + 7(1-p')$$

Thus, 1 chooses C if $p' > 0.5$ and N if $p' < 0.5$.

**Subcase 1**: If $p' < 0.5$, $BR_1(N) = N$. Then both type I and II have an incentive to deviate.

**Subcase 2:** If $p' > 0.5$; $BR_1(N) = C$. In this case both type I and type II have an incentive to deviate.

**Case 2:** By definition of pooling equilibrium, the posterior belief is the same as the prior belief on the equilibrium path: $\mu(I|N) = p$: If the received signals is not N, but C, then $\mu(I|C) = p'$. Consider first the best response of player 1. Suppose that player 1 observes N, then

$$\pi_1(C) = 7p + 5(1-p)$$

$$\pi(N) = 5p + 7(1-p)$$

Since $p > 0.5$, player 1 chooses C. Now, suppose that player 1 observes C. Then

$$\pi_1(C) = -2(1-p')$$

$$\pi(N) = -2p'$$

Thus, 1 chooses N if $p' < 0.5$ and C if $p' \geq 0.5$. Again, there are two subcases.

**Subcase 1:** If $p' < 0.5$, $BR_1(C) = N$. Type I has an incentive to deviate, since by playing C she increases her payoff from -2 to 7.

**Subcase 2:** If $p' > 0.5$; $BR_1(C) = C$. Again, type I has an incentive to deviate, since by playing C she increases her payoff from -2 to 0.

Thus, there is no pooling equilibrium.

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In what follows, we will investigate mixed strategy PBE. Suppose that $m(C)$ and $m(N)$ are the conditional probability of player 1 choosing C if he has, respectively, seen signals C and N. Suppose that $q^I$ and $q^{II}$ are the probabilities of the two types choosing C.

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**iii.** Show that the posterior probability of type I, conditional on player 1 seeing signal C is:

$$p\left(C\right)=\frac{p.q_{I}}{p.q_{I}+\left(1-p\right)q_{II}}$$

Define the analogous posterior probability conditional on seeing N. Call it $p(N)$.

**Solution:** By definition of Bayesian updating:

$$p\left(C\right)=\frac{pq^{I}}{pq^{I}+\left(1-p\right)q^{II}}$$

$$p\left(N\right)=\frac{p\left(1-q^{I}\right)}{p\left(1-q^{I}\right)+\left(1-p\right)\left(1-q^{II}\right)}$$

**iv.** Consider player 1's best response to signal C. Show that player 1 will himself mix, i.e. $m(C)\in(0,1)$ if and only if $p(C)=0.5$. Show that the same conclusion holds for the signal N.

**Solution:** Suppose that the signal is C. Then the payoffs of player 1 from playing each pure strategy is:

$$\pi_{1}\left(C\right)=-2\left(1-p\left(C\right)\right)$$

$$\pi\left(N\right)=-2p\left(C\right)$$

To play a mixed strategy, player 1 must be indifferent between the payoffs offered by these pure strategies, given the strategy profile of types I and II. This condition is satisfied when $p(C)=0.5$. Next, suppose that the signal is N. Then the payoffs of player 1 from playing each pure strategy is:

$$\pi_{1}\left(C\right)=7p\left(N\right)+5\left(1-p\left(N\right)\right)$$

$$\pi_{1}\left(N\right)=5p\left(N\right)+7\left(1-p\left(N\right)\right)$$

Again, player 1 mixes if $p(N)=0.5$.

**v.** Using iv, show that after seeing one of the signals C or N player 1 does not mix, i.e. m(C)=0 or 1 (m(N)=0 or 1).

**Solution:** From above, we have that player 1 mixes only if

$$0.5=\frac{pq^{I}}{pq^{I}+\left(1-p\right)q^{II}}$$

$$0.5=\frac{p\left(1-q^{I}\right)}{p\left(1-q^{I}\right)+\left(1-p\right)\left(1-q^{II}\right)}$$

Solving this system of equations we arrive at p=0.5, which contradicts the assumption that $p>0.5$.

**vi.**Use this fact to argue that in a PBE exactly one type will not mix either, i.e. $q^{I}=0$ or 1 (or $q^{II}=0$ or 1).

**Solution:** By contradiction. Type I plays a mixed strategy if

$$7\left(1-m\left(C\right)\right)=-2m\left(N\right)+5\left(1-m\left(N\right)\right)$$

implying that

$$m\left(C\right)=m\left(N\right)+\frac{2}{7}$$

Type II mixes if

$$-2m\left(C\right)+5\left(1-m\left(C\right)\right)=5m\left(N\right)+7\left(1-m\left(N\right)\right)$$

implying that

$$\frac{2}{7}-\frac{2}{7}m\left(N\right)=m\left(C\right)$$

These two conditions imply that $m(N)=0$, a contradiction.

# 3 A Simple Signalling Game 2

Consider the same setting as in the preceding exercise, but now let the probability of playing against type I is $p<0.5$.

1. Find all separating PBE.

   Solution: There are two possible separating equilibria:

| | Case 1 | Case 2 |
|---|---|---|
| Type I | C | N |
| Type II | N | C |

   **Case 1:** By definition of separating equilibrium,

$$\mu\left(I|C\right)=1 \text{ and } \mu\left(II|N\right)=1$$

   so player 1 knows exactly the correct payoff matrix and the relevant payoff column, C for type I and N for type II. The best response of 1 are $BR_1(C)=C$ and $BR_1(N)=N$. What remains is to check whether type I or type II has an incentive to deviate. If type I does not deviate, then her payoff from C is 0. If she deviates by playing N player 1 will mistake her to be type II and will play N, so type I will get a payoff of 5. Consequently, type I has an incentive to deviate. Thus, there is no separating PBE in which the types send signals (C,N).

   **Case 2:** By definition of separating equilibrium,

$$\mu\left(I|N\right)=1 \text{ and } \mu\left(II|C\right)=1$$

   Also $BR_1(N)=C$ and $BR_1(C)=N$. Clearly type I has an incentive to deviate to N, so there is no separating equilibrium in which the types send signals (N,C).

2. Find all the pooling pure strategy equilibria.

   Solution: There are two possible pooling equilibria:

| | Case 1 | Case 2 |
|---|---|---|
| Type I | C | N |
| Type II | C | N |

   **Case 1:** By definition of pooling equilibrium, the posterior belief is the same as the prior belief on the equilibrium path: $\mu(I|C) = p$: If the received signals is not C, but N, then $\mu(I|N) = p'$. Consider first the best response of player 1. Suppose that player 1 observes C, then

$$\pi_1(C) = -2(1-p)$$

$$\pi_1(N) = -2p$$

   Since $p < 0.5$, 1 will play $N$. Now, suppose that player 1 observes N (which may happen if one of the types deviates). Then

$$\pi_1(C) = 7p' + 5(1-p')$$

$$\pi(N) = 5p' + 7(1-p')$$

Thus, 1 chooses $N$ if $p' < 0.5$ and $C$ if $p' > 0.5$.

**Subcase 1**: If $p' < 0.5$, $BR_1(N) = N$. Then type I has no incentive to deviate. However type II has, since by playing N she increases her payoff from 5 to 7 .

**Subcase 2:** If $p' > 0.5$; $BR_1(N) = C$. In this case neither type I nor type II have an incentive to deviate from the pooling equilibrium.

Thus, $(N, C, C)$ define the strategies of a pooling equilibrium of the game.

**Case 2:** By definition of pooling equilibrium, the posterior belief is the same as the prior belief on the equilibrium path: $\mu(I|N) = p$: If the received signals is not N, but C, then $\mu(I|C) = p'$. Consider first the best response of player 1. Suppose that player 1 observes N, then

$$\pi_1(C) = 7p + 5(1-p)$$

$$\pi(N) = 5p + 7(1-p)$$

Since $p < 0.5$, player 1 chooses C. Now, suppose that player 1 observes C. Then

$$\pi_1(C) = -2(1-p')$$

$$\pi(N) = -2p'$$

Thus, 1 chooses N if $p' < 0.5$ and C if $p' \geq 0.5$. Again, there are two subcases.

**Subcase 1:** If $p' < 0.5$, $BR_1(C) = N$. Type I has an incentive to deviate, since by playing C she increases her payoff from -2 to 7.

**Subcase 2:** If $p' > 0.5$; $BR_1(C) = C$. Again, type I has an incentive to deviate, since by playing C she increases her payoff from -2 to 0.

Thus, there is no pooling equilibrium in this case.