Ecole Polytechnique
ECO 560 Economics of Contracts

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Week 1: Class Notes

1 Insurance

Suppose that a DM is risk-averse with wealth y and runs a risk of a loss of d with probability p. The probability of loss is p. The DM's problem is to find the optimal units of insurance, α, where the price of one unit of insurance is c, and the insurer pays back α in case of loss. u is increasing, twice differentiable, and concave. The problem of the DM is:

$$\max_{\alpha \geq 0} (1-p)u(y-\alpha c) + pu(y-\alpha c - d + \alpha)$$

1. Find the first order conditions corresponding to this problem. Simplify the corresponding conditions to one inequality and work with it for the rest of the problem.

Hint: These are the same conditions as in the case of equal constraints, along with three more:

$$\lambda.\alpha = 0, \alpha \geq 0, \lambda \geq 0$$

where λ is the Lagrange multiplier.

Solution: By the Kuhn-Tucker first order conditions, we have that at the optimal level of insurance, α:

$$-(1-p)u'(y-\alpha c)c + (1-c)pu'(y-\alpha c - d + \alpha) - \lambda = 0$$

$$\lambda.\alpha = 0, \alpha \geq 0, \lambda \geq 0$$

or alternatively, we have:

$$-(1-p)u'(y-\alpha c)c + (1-c)pu'(y-\alpha c-d+\alpha) \leq 0$$

2. Suppose that the insurance is actuarially fair, c $= p$. Simplify the conditions.

Solution: The conditions simplify to:

$$-(1-p)u'(y-\alpha p)p + (1-p)pu'(y-\alpha p-d+\alpha) \leq 0$$

so, the conditions simplify to:

$$u'(y-\alpha p-d+\alpha) - u'(y-\alpha p) \leq 0$$

3. Prove that the DM insures completely.

Solution: Observe that $u'(y-d) > u'(y)$, so $\alpha > 0$. Thus:

$$u'(y-\alpha p-d+\alpha) = u'(y-\alpha p)$$

Since u'(.) is monotone, we have that:

$$y-\alpha p-d+\alpha = y-\alpha p$$

which implies that $\alpha = d$. That is, the DM ensures completely.

4. Now, suppose that $c \geq p$. Prove that the DM would not buy full insurance.

Solution: Suppose that the optimal level of insurance is $\alpha = d$. The first order condition implies that the following must be true:

$$-(1-p)u'(y-dc)c + (1-c)pu'(y-dc) = 0$$

However, we have:

$$-(1-p)u'(y-dc)c + (1-c)pu'(y-dc) < 0$$

so the optimal level of insurance is $\alpha < d$.

2 Demand for a Risky Asset

Consider a risk-averse DM who can invest her wealth Y in any of the following two assets. R_1 pays-off r > 1 for each dollar invested in any state of the world. R_2 is a risky asset that pays θ_i with probability $p(\theta_i) > 0$[1]. $p(\theta)$ is a simple lottery with finite number of states with payoffs $\theta_1, ..., \theta_n$. $E(\theta) > r$. There exists at least one state i in which $p(\theta_i) > 0$ and $\theta_i < r$. The decision to allocate Y between the two assets is represented by the following program:

$$\begin{aligned} \max_{\alpha} U &= \sum_{i=1}^{n} p(\theta_i) u(\alpha Y \theta_i + (1-\alpha) Y r) \\ s.t.\ \ Y &= \alpha Y + (1-\alpha) Y, \text{ i.e.} \\ \alpha &\geq 0 \text{ and } \alpha \leq 1 \end{aligned}$$

Observe that U is continuous function in α, defined on a compact set, so the function U has a maximum on the set by the Weirstrass Theorem. We will show that DM invests $\alpha > 0$ in the risky asset.

1. Suppose not: suppose that the optimal level of investment in the risky asset R_2 is $\alpha = 0$. Find the first-order condition at the optimum when $\alpha = 0$.

Solution: At the optimum α the following first-order condition is supposed to hold:

$$\sum_{i=1}^{n} p(\theta_i) u'(Y(\alpha^*(\theta_i - r) + r))(\theta_i - r) + \lambda_1 - \lambda_2 = 0$$

$$\lambda_1 . \alpha = 0,\ \lambda_1 \geq 0,\ \alpha \geq 0$$

$$\lambda_2 . \alpha = 0,\ \lambda_2 \geq 0,\ \alpha \leq 1$$

at $\alpha = 0$ these conditions simplify to:

$$\sum_{i=1}^{n} p(\theta_i) u'(Yr))(\theta_i - r) \leq 0$$

2. Use the first-order condition to prove the result.

Solution: Since $\theta_i - r > 0$, this condition cannot be satisfied, a contradiction. Thus, the agent holds positive amount of the risky asset. ■

[1] Note that $p(\theta_i)$ here means the probability that outcome θ_i realizes; it does NOT mean that p is a function of θ_i.

Remark: The only way this condition may hold is if $E(\theta_i - r) \leq 0$. Indeed, if $E(\theta_i - r) \leq 0$, then the optimal solution is $\alpha = 0$; a risk-averse individual will never invest in an asset with an expected return lower than a risk-free asset.

3 Perfect Discrimination

Consider a seller and a buyer and suppose that the seller is perfectly informed about the characteristics of the buyer. The buyer can be of two types θ_1 and θ_2, $\theta_1 < \theta_2$. The probability of type θ_1 is π. The utility of type i is

$$\theta_i v(q_i) - t_i$$

where $v(q)$ is increasing. The seller offers to the buyer contracts that consist of quantity and price and the buyer decides whether to accept or reject the offer. The cost of producing quantity q is cq. If she rejects it, she receives reservation utility of 0. The seller is a risk-neutral monopolist.

1. State the problem of the principal.

Solution: The problem of the principal is:

$$\max_{(q_i,t_i)} \pi (t_1 - cq_1) + (1-\pi)(t_2 - cq_2)$$

subject to:

$$\theta_1 v(q_1) - t_1 \geq 0$$
$$\theta_2 v(q_2) - t_2 \geq 0$$

2. Prove that the individual rationality constraints bind and use them to simplify the problem of the principal.

Solution: Clearly, the constraints are going to bind. From them, we can simplify the problem of the principal:

$$\max_{(q_i,t_i)} \pi (\theta_1 v(q_1) - cq_1) + (1-\pi)(\theta_2 v(q_2) - cq_2)$$

3. Find conditions that (implicitly) determine the solution of the problem.

Solution: The fist order conditions of this problem lead to the following conditions that allow us to find the optimal q_i :

$$\theta_i v'(q_i) = 1$$

4. Interpret the results. Discuss examples from real life where perfect discrimination is possible.

Solution: Thus, the optimal quantity for type 1 is less than the optimal quantity for type 2. Also, note that the seller extracts all the surplus from the relation with each type. This comes from its monopolist power and the fact that the seller is allowed to make take-it-or-leave it offers.

4 Linear Pricing with Unknown Types

Consider the setting from above. However, assume that the seller does not know the type of the agents and by law is restricted to linear pricing. That is $t_i = p.q_i$. Suppose that it is optimal for the seller to sell to both types: that is, the individual rationality constraints are satisfied.

1. State the problem of the buyer for a given p and find the first order condition.

Solution: For a given p, the problem of the buyer of type i is:

$$\max_{q_i} \theta_i v(q_i) - pq_i$$

which implies the first order condition:

$$\theta_i v'(q_i) = p$$

2. Solve the condition for q_i as a function of p.

Solution: This condition can be inverted to determine the demand of type i :

$$q_i = D(p, \theta_i) = d_i(p)$$

3. Use the solution to simplify the problem of the principal.

Solution: We can use the demand function for type i to simplify the problem of the seller.

$$\max_{(p)} \pi((p-c) d_1(p)) + (1-\pi)((p-c) d_2(p))$$

4. Use the first-order conditions to find the profit-maximizing price. Let aggregate demand under price p be $D(p)$.

Solution: From the first order conditions, we have that p is the solution of:

$$\underbrace{(\pi d_1(p) + (1-\pi) d_2(p))}_{D(p)} + (p-c) \underbrace{(\pi d_1'(p) + (1-\pi) d_2'(p))}_{D'(p)} = 0$$

which can be rewritten as:

$$p_m = c - \frac{D'(p)}{D(p)}$$

Note that $D(p)$ is a decreasing positive function. Thus, the price that the monopolist will charge is higher than c.

5. Suppose that the firm decides to serve only the high type. Find the profit-maximizing price.

Solution: Note that conceptually, the presence of asymmetric information does not change the monopolist's problem and the solution. This is the case, however, when the monopolist finds it optimal to serve all types. It may very well be the case that the monopolist would like to exclude the low type and charge

$$p = \frac{\theta_2 v(q_2)}{q_2}$$

In this way, the monopolist loses the profits from the low type but gains all the surplus from the high types.

5 Two-Part Tariffs

Consider the setting from above. However, assume that the seller does not know the type of the agents and by law is restricted to a single two-part tariff: a fixed payment and a tariff rate p. Thus, the total amount that the buyer pays for quantity q is $t_i = f + p.q$.

1. Which individual rationality constraints will bind?

Solution: Suppose that the seller would like to serve both types of customers. Then the individual rationality constraints must be satisfied:

$$\theta_1 v(q) - pq - f \geq 0$$

$$\theta_2 v(q) - pq - f \geq 0$$

If the first is satisfied, the second constraint will always be satisfied. Furthermore, profit-maximization requires that:

$$\theta_1 v(q) - pq = f$$

2. State the problem of the buyer and find conditions that allow you to solve for the optimal demand.

Solution: The problem of the buyer of type i is then

$$\max_{q} \theta_i v(q) - pq - f$$

The first order conditions imply that:

$$\theta_i v'(q_i) = p$$

This condition can be inverted to determine the demand of type i :

$$q_i = D(p, \theta_i) = d_i(p)$$

3. Let the optimal quantity as a function of p be $d_i(p)$ for type i. Calculate the surplus of the low type and find the optimal payment that the seller would request.

Solution: The surplus of type 1 is then:

$$s_1(p) = \theta_1 v(d_1(p)) - pd_1(p)$$

From above, we know that the seller would extract all surplus from the low type by setting the fixed payment equal to $s_1(p)$.

4. Let the aggregate demand under price p be $D(p)$. Use the above results to simplify the problem of the seller and find conditions that allow you to find the optimal price.

Solution: Thus, the problem of the seller becomes:

$$\max_{(p)} (p - c) D(p) + s_1(p)$$

From the first order conditions, we have that p is the solution of:

$$p_{tariff} = c - \frac{D(p) + s_1'(p)}{D'(p)}$$

By the Envelope Theorem,

$$s_1'(p) = -d_1(p)$$

Thus, $D(p) + s_1'(p)$ is strictly positive.