Ecole Polytechnique
ECO 560 Economics of Contracts

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Lecture 2: Adverse Selection

1 Introduction

In this lecture we cover the basic screening models. These models are also known as models of adverse selection and optimal nonlinear pricing. We start with a model of screening with two types because this allows us to appreciate the features of the model in a friendly, noncomplicated environment. Then, we cover the general model of screening with a continuum of types. Both models have been based on Mussa and Rosen (1978). I consider only deterministic contracts here: the proof that the optimal menu of contracts is deterministic under some general conditions is left for Problem Set 1.

2 Screening: Model with Two Types

2.1 Setting

Consider a seller who faces two types of customers: with probability π he faces customer of type θ_1 and $(1-\pi)$ customer of type θ_2. Let q be the quality purchased and t be the transfer made from the buyer to the seller. The buyer's utility is $U(\theta_i,(q,t)) = \theta_i q - t$.The seller's cost is C($q$) , where C($q$) is convex, twice differentiable, C'(0) = 0 and C'(∞) = ∞ The seller's profits from serving a customer are $t - C(q)$.

To find the optimal pair of contracts, we solve

$$\max_{t_1,q_1,t_2,q_2} \{\pi (t_1 - C(q_1)) + (1-\pi)(t_2 - C(q_2))\}$$

s.t.

$$\theta_1 q_1 - t_1 \geq \theta_1 q_2 - t_2 \quad (IC1)$$
$$\theta_2 q_2 - t_2 \geq \theta_2 q_1 - t_1 \quad (IC2)$$
$$\theta_1 q_1 - t_1 \geq \underline{u} \quad (IR1)$$
$$\theta_2 q_2 - t_2 \geq \underline{u} \quad (IR2)$$

Normalizing the reservation utility to 0, we have problem **(P)**:

$$\max_{t_1,q_1,t_2,q_2} \{\pi (t_1 - C(q_1)) + (1-\pi)(t_2 - C(q_2))\}$$

s.t.

$$\theta_1 q_1 - t_1 \geq \theta_1 q_2 - t_2 \quad (IC1)$$
$$\theta_2 q_2 - t_2 \geq \theta_2 q_1 - t_1 \quad (IC2)$$
$$\theta_1 q_1 - t_1 \geq 0 \quad (IR1)$$
$$\theta_2 q_2 - t_2 \geq 0 \quad (IR2)$$

2.2 Benchmark Case: Types are known

When the Principal knows the type of the Agent, he can extract all the surplus from the relation:

$$t_1 = \theta_1 q_1$$
$$t_2 = \theta_2 q_2$$

Since the Principal can perfectly identify the customer, the Principal's problem for customer i becomes:

$$\max_{q_i} \theta_i q_i - C(q_i)$$

which of course implies that the optimal q_i^* is the solution to

$$\theta_i - C'(q_i) = 0.$$

However, if the Principal does not know the type of the buyer, then the contracts above would not help the Principal find out what is the type of the customer. Note that from above:

$$0 = \theta_1 q_1^* - t_1 < \theta_2 q_1^* - t_1$$

$$0 = \theta_2 q_2^* - t_2 < \theta_2 q_1^* - t_1$$

so type 2 would have an incentive to present herself as type 1.

2.3 Solving the Model 1

Proposition 1: At the optimum for program **(P)**:

1. IR 1 binds, so $t_1 = \theta_1 q_1$.
2. IC 2 binds, so $t_2 - t_1 = \theta_2(q_2 - q_1)$.
3. $q_2 \geq q_1$
4. q_2 is equal to the efficient quantity (no distortion on top).
5. IC 1 and IR 2 are redundant.

   Proof: The problem becomes, after the normalization of $\underline{u} = 0$.

$$\max_{t_1, q_1, t_2, q_2} \{\pi (t_1 - C(q_1)) + (1 - \pi)(t_2 - C(q_2))\}$$

s.t.

$$\theta_1 q_1 - t_1 \geq \theta_1 q_2 - t_2 \quad (IC1)$$

$$\theta_2 q_2 - t_2 \geq \theta_2 q_1 - t_1 \quad (IC2)$$

$$\theta_1 q_1 - t_1 \geq 0 \quad (IR1)$$

$$\theta_2 q_2 - t_2 \geq 0 \quad (IR2)$$

The revalation principle stipulates that if an allocation (q, t) can be implemented through a menu of contracts, then it can also be implemented through a menu of contracts that includes only one offer for each type. Moreover, the solution is such that each type finds it in her interest to choose the offer associated with her true type. Show that if the seller is maximizing profits when types are unknown, it must be the case that he offers:

1. **Proof:** From IC 2:

$$\theta_2 q_2 - t_2 \geq \theta_2 q_1 - t_1$$

Since $\theta_1 < \theta_2$

$$\theta_2 q_1 - t_1 > \theta_1 q_1 - t_1$$

Suppose that IR 1 does not bind

$$\theta_1 q_1 - t_1 > 0$$

From above this implies that

$$\theta_2 q_2 - t_2 > 0$$

Consequently, we can increase t_1 and t_2 by the same amount, preserve the choice of the type, and increase the profits of the Principal. Thus, the original menu of contracts is not optimal, which contradicts the assumptions. Thus, IR 1 binds and $t_1 = \theta_1 q_1$.

2. **Proof:** Suppose in contradiction that IC 2 does not bind, i.e.

$$\theta_2 q_2 - t_2 > \theta_2 q_1 - t_1$$

By property 1 and since $\theta_1 < \theta_2$:

$$\theta_2 q_1 - t_1 > \theta_1 q_1 - t_1 = 0$$

Again, we can increase t_2 by a small amount to increase profits, while preserving incentive compatibility and individual rationality. Again, this implies a contradiction with the assumption that the proposed menu of contracts is optimal. Thus, IC 2 binds so

$$t_2 - t_1 = \theta_2(q_2 - q_1).$$

By property 1 then, we can eliminate

$$t_2 = \theta_2(q_2 - q_1) + \theta_1 q_1$$

3. **Proof:** Draw the indifference curves for types θ_1 and θ_2 that go through (q_1, t_1) in a $q - t$ coordinate system and determine the preferred sets for each type. The previous part indicates that (q_1, t_1) must belong to the indifference curve that goes through the origin giving the IR

level of utility 0. Further, the optimal bundle for θ_2 must belong to the indifference curve for the θ_2 that goes through (q_1, t_1). Since both IC_2 and IR_1 need to be satisfied at the optimum, the optimal offer for the high type must belong to her indifference curve that goes through (q_1, t_1). We will show that $q_2 > q_1$. Note that under t_1 and t_2 derived in 1 and 2, such that IR_1 and IC_2 are satisfied, the first best quantities $\left(q_1^{FB}, q_2^{FB}\right)$ can be achieved even when types are not observed. Note that $\left(q_1^{FB}, t_1^{FB}\right)$ satisfied IR_1. To ensure that incentive compatibility is also satisfied, we need to draw an indifference curve through $\left(q_1^{FB}, t_1^{FB}\right)$. This indifference curve intersects the vertical line $q = q_2^{FB}$. The transfer is

$$t_2 = \theta_2(q_2^{FB} - q_1^{FB}) + \theta_1 q_1^{FB}$$

The question is whether this schedule is still optimal. $\left(q_1^{FB}, q_2^{FB}\right)$ are the maximizers of the profit function when types are known:

$$\pi\theta_1 q_1 + (1-\pi)\,\theta_2 q_2 - \pi C\,(q_1) - (1-\pi)\,C\,(q_2)$$

where π is the probability of the low type. However, the objective function of the Principal when types are **not known** is:

$$\pi\theta_1 q_1 + (1-\pi)\,\theta_2 q_2 - (1-\pi)\,(\theta_2 - \theta_1)\,q_1 - \pi C\,(q_1) - (1-\pi)\,C\,(q_2)$$

That is, there is the term $-(1-\pi)\,(\theta_2 - \theta_1)\,q_1$ which implies that the optimal level for the low type when types are not knwon is less than q_1^{FB}. Thus, we have that the optimal (q_2, t_2) is such that $q_2 > q_1$. By the single-crossing property, it belongs to the set of bundles (q, t) which are weakly prefered by the high type but are dominated by (q_1, t_1) from the perspective of θ_1.

4. **Proof:** The result follows from the first-order conditions for optimality when types are not known.

5. **Proof:** From 3, IC_1 holds for the optimal contract. IR_2 also holds since

$$\theta_1 q_1 - t_1 < \theta_2 q_1 - t_1 = \theta_2 q_2 - t_2$$

'Purely Algebraic' solution

Proof: From IC 2:

$$\theta_2 q_2 - t_2 \geq \theta_2 q_1 - t_1$$

Since $\theta_1 < \theta_2$

$$\theta_2 q_1 - t_1 > \theta_1 q_1 - t_1$$

Suppose that IR 1 does not bind

$$\theta_1 q_1 - t_1 > 0$$

From above this implies that

$$\theta_2 q_2 - t_2 > 0$$

Consequently, we can increase t_1 and t_2 by the same amount, preserve the choice of the type, and increase the profits of the Principal. Thus, the original menu of contracts is not optimal, which contradicts the assumptions. Thus, IR 1 binds.

Suppose in contradiction that IC 2 does not bind, i.e.

$$\theta_2 q_2 - t_2 > \theta_2 q_1 - t_1$$

By property 1 and since $\theta_1 < \theta_2$:

$$\theta_2 q_1 - t_1 > \theta_1 q_1 - t_1 = 0$$

Again, we can increase t_2 by a small amount to increase profits, while preserving incentive compatibility and individual rationality. Again, this implies a contradiction with the assumption that the proposed menu of contracts is optimal. Thus, IC 2 binds so

$$t_2 - t_1 = \theta_2(q_2 - q_1).$$

By property 1 then, we can eliminate t_1

$$t_2 = \theta_2(q_2 - q_1) + \theta_1 q_1$$

Let us add IC 1 and IC 2. The transfers cancel out and we are left with:

$$\theta_2 \left(q_2 - q_1\right) \geq \theta_1 \left(q_2 - q_1\right)$$

$$\left(\theta_2 - \theta_1\right)\left(q_2 - q_1\right) \geq 0$$

Since $\theta_2 > \theta_1$, $q_2 - q_1 \geq 0$.

Since both IC_2 and IR_1 need to be satisfied at the optimum:

$$t_1 = \theta_1 q_1$$

$$t_2 = \theta_2(q_2 - q_1) + \theta_1 q_1$$

Note that under t_1 and t_2 the first best quantities $\left(q_1^{FB}, q_2^{FB}\right)$ can be achieved even when types are not observed: the question is whether these quantities are still optimal. In particular, IR_1 is satisfied for the first best $\left(q_1^{FB}, \theta_1 q_1^{FB}\right)$. Next, observe that

$$(q_2^{FB}, \theta_2(q_2^{FB} - q_1^{FB}) + \theta_1 q_1^{FB})$$

satisfies IC_2 with an equality. We will show that the optimal q_1 when types are unknown is less than q_1^{FB}. $\left(q_1^{FB}, q_2^{FB}\right)$ are the maximizers of the following profit function when types are known:

$$\pi\theta_1 q_1 + (1-\pi)\,\theta_2 q_2 - \pi C\,(q_1) - (1-\pi)\,C\,(q_2)$$

Using IR_1 and IC_2, we can simplify the objective function of the Principal when types are not known to:

$$\pi\theta_1 q_1 + (1-\pi)\,\theta_2 q_2 - (1-\pi)\,(\theta_2 - \theta_1)\,q_1 - \pi C\,(q_1) - (1-\pi)\,C\,(q_2)$$

Consider the unconstrained optimization of this function with respect to (q_1, q_2). Note that, we have the term $-(1-\pi)\,(\theta_2 - \theta_1)\,q_1$ which implies that the solution for q_1 is less than q_1^{FB}. At the same time, the solution for q_2 is equal to q_2^{FB}. We will show that the solution of this problem, (q_1^*, q_2^*), satisfies IC_1 and IR_2. Rewriting IC 1

$$\theta_1(q_2 - q_1) \leq t_2 - t_1$$

From property 2, we have that

$$t_2 - t_1 = \theta_2(q_2 - q_1)$$

From above $(q_2^* - q_1^*) \geq 0$. Thus, IC 1 is satisfied at (q_1^*, q_2^*).

Furthermore, from IC 2:

$$\theta_2 q_2 - t_2 = \theta_2 q_1 - t_1$$

Since $\theta_1 < \theta_2$

$$\theta_2 q_1 - t_1 > \theta_1 q_1 - t_1 = 0$$

Thus, if IR 1 holds, i.e.

$$\theta_1 q_1^* - t_1 = 0$$

implies that

$$\theta_2 q_2^* - t_2 > 0.$$

Suppose in contradiction that the optimal q_2 is different from the efficient quantity that we found in the benchmark case. From the properties of the cost function, this implies that

$$C'(q_2) < \theta_2$$

Now consider an alternative menu of contracts, where $\varepsilon > 0$ is sufficiently small: $(q_1, t_1), (q_2' = q_2 + \varepsilon, t_2' = t_2 + \varepsilon\theta_2)$. Note that

$$\theta_2 q_2' - t_2' = \theta_2 q_2 - t_2$$

so the IC 2 and IR 2 is satisfied as before. Since

$$\theta_1 q_2' - t_2' = \theta_1 q_2 - t_2 - \varepsilon(\theta_2 - \theta_1)$$

so that IC 1 is also satisfied. However,

$$t_2' - C(q_2') \simeq t_2 - C(q_2) + \varepsilon(\theta_2 - C'(q_2))$$

Consequently, the profits under the new menu of contracts are higher than the original, a contradiction. ■

2.4 Solving the Model 2

By Proposition 1, we can modify the problem as follows:

$$\max_{t_1, q_1, t_2, q_2} \{\pi(t_1 - C(q_1)) + (1 - \pi)(t_2 - C(q_2))\}$$

s.t.

$$\theta_2 q_2 - t_2 = \theta_2 q_1 - t_1 \quad (IC2)$$

$$\theta_1 q_1 - t_1 = \underline{u} \ (IR1)$$

where

$$t_1 = \theta_1 q_1$$

$$t_2 = \theta_2(q_2 - q_1) + \theta_1 q_1$$

Substituting the equality constraints into the objective function, we have to solve the following problem of unconstrained optimization:

$$\max_{q_1,q_2} \{\pi (\theta_1 q_1 - C(q_1)) + (1-\pi)(\theta_2(q_2 - q_1) + \theta_1 q_1 - C(q_2))\}$$

Reorganizing terms, we have:

$$\max_{q_1,q_2} \{\pi (\theta_1 q_1 - C(q_1)) + (1-\pi)(\theta_2 q_2 - C(q_2) - (\theta_2 - \theta_1) q_1)\}$$

Thus, the first term in the objective function is equal to the full surplus from the market exchange with type 1, while the second term is equal to less than the full surplus of the market exchange with type 2. The term $(\theta_2 - \theta_1) q_1$ is the information rent that type 2 obtains in return for revealing her type.

Solution. From above, we already know that the optimal q_2 is the efficient level. This can be seen again from the FOC for q_2:

$$C'(q_2) = \theta_2$$

The FOC for q_1 is:

$$\theta_1 - \frac{1-\pi}{\pi}(\theta_2 - \theta_1) = C'(q_1)$$

Since

$$\theta_1 - \frac{1-\pi}{\pi}(\theta_2 - \theta_1) < \theta_1$$

so type 1 receives less than the efficient level of commodity 1.

2.5 Properties of the solution:

- The high type gets an efficient allocation of the commodity.
- All other types get subefficient amounts of the commodity.
- All types, except for the lowest type, get positive information rents, which increases with the type.
- The lowest type gets a zero surplus.
- In general little can be said about the shape of $T(q)$ schedule.

3 Screening: General Model

3.1 Setting

Consider a Principal who faces a continuum of types $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ with density $f(\theta)$. The buyer's utility of type θ from declaring type θ' is $U(\theta, \theta') = \theta q(\theta') - t(\theta')$. This utility function satisfies the so called single crossing property

$$\frac{\partial^2 U}{\partial q \partial \theta} \cdot \frac{\partial q}{\partial \theta} > 0$$

Alternative notation for this derivative that we will use for brevity is U_{12}. The single-crossing property is must be satisfied when we use a more general formulation of the Agent's utility. Let $q(\theta)$ be the quality purchased by θ and $t(\theta)$ be the transfer made from the Agent θ to the Principal. The Principal's cost is C(q), where C(q) is convex, twice differentiable, C'(0) = 0 and C'(∞) = ∞ The Principal's profits from serving a customer θ are $t(\theta) - C(q(\theta))$.

The problem of the Principal is:

$$\max_{t(\theta), q(\theta)} \int_{\underline{\theta}}^{\overline{\theta}} (t(\theta) - C(q(\theta))) f(\theta) d\theta$$

s.t. for all θ, θ'

$$\theta q(\theta) - t(\theta) \geq \theta q(\theta') - t(\theta') \quad (IC)$$

$$\theta q(\theta) - t(\theta) \geq 0 \quad (IR)$$

3.2 Preliminaries

Observation: If

$$\theta q(\underline{\theta}) - t(\underline{\theta}) \geq 0 \quad (IR)$$

then IR is satisfied for all θ.

IC conditions look bad. We will try to show that it is sufficient that they are satisfied locally.

Lemma: IC holds if and only if (i) $q(\theta)$ is increasing and (ii) for all θ

$$\frac{\partial U}{\partial \theta'}|_{\theta'=\theta} = U_2(\theta, \theta) = 0 \, (local \ IC)$$

Proof: See notes for PC 2.

Thus, we restate the problem as follows:

$$\max_{t(\theta),q(\theta)} \int_{\underline{\theta}}^{\overline{\theta}} \left(t\left(\theta\right) - C\left(\theta\right)\right) f\left(\theta\right) d\theta$$

s.t.

$$\theta q\left(\underline{\theta}\right) - t\left(\underline{\theta}\right) \geq 0 \;\; \left(IR\left(\underline{\theta}\right)\right)$$

$$U_2\left(\theta,\theta\right) = 0$$

$$q\left(\theta\right) \text{ increasing}$$

Implications of local IC:

$$U_2\left(\theta,\theta\right) = 0 \quad <=>$$

$$\frac{dU\left(\theta,\theta\right)}{d\theta} = U_1\left(\theta,\theta\right) = q\left(\theta\right)$$

$$U\left(\theta,\theta\right) - U\left(\underline{\theta},\underline{\theta}\right) = \int_{\underline{\theta}}^{\theta} q\left(x\right) dx \quad \left(FT \text{ of } C\right)$$

$$U\left(\theta,\theta\right) = \underline{\theta} q\left(\underline{\theta}\right) - t\left(\underline{\theta}\right) + \int_{\underline{\theta}}^{\theta} q\left(x\right) dx$$

$\int_{\underline{\theta}}^{\theta} q\left(x\right) dx$ is the information rent of type θ.

Which allows us to simplify the problem even further.

3.3 Solving the Model

By the results in the previous subsection, the problem of the Principal is equivalent to:

$$\max_{t(\theta),q(\theta)} \int_{\underline{\theta}}^{\overline{\theta}} \left(t\left(\theta\right) - C\left(q\left(\theta\right)\right)\right) f\left(\theta\right) d\theta$$

s.t.

$$\theta q\left(\underline{\theta}\right) - t\left(\underline{\theta}\right) \geq 0 \;\; \left(IR\left(\underline{\theta}\right)\right)$$

$$U\left(\theta,\theta\right) = \underline{\theta} q\left(\underline{\theta}\right) - t\left(\underline{\theta}\right) + \int_{\underline{\theta}}^{\theta} q\left(x\right) dx \quad \left(IC\right)$$

$$q\left(\theta\right) \text{ increasing}$$

Solution:
Part A: The IR constraint bind. so

$$\theta q(\theta) - t(\theta) = \int_{\underline{\theta}}^{\theta} q(x)\,dx \;\; =>$$

$$t(\theta) = \theta q(\theta) - \int_{\underline{\theta}}^{\theta} q(x)\,dx$$

We will substitute this expression for the transfer into the objective function. For the moment, we will ignore the monotonicity requirement and check under what conditions it is satisfied at the end. The program becomes:

$$\max_{q(\theta)} \int_{\underline{\theta}}^{\overline{\theta}} \left(\theta q(\theta) - \int_{\underline{\theta}}^{\theta} q(x)\,dx - C(q(\theta)) \right) f(\theta)\,d\theta$$

which can be rewritten as:

$$\max_{q(\theta)} \int_{\underline{\theta}}^{\overline{\theta}} (\theta q(\theta) - C(q(\theta)))\, f(\theta)\,d\theta - \int_{\underline{\theta}}^{\overline{\theta}} \underbrace{\int_{\underline{\theta}}^{\theta} q(x)\,dx}_{u} \underbrace{f(\theta)\,d\theta}_{dv}$$

Next, we integrate the objective function by parts:

$$\int_{\underline{\theta}}^{\overline{\theta}} (\theta q(\theta) - C(q(\theta)))\, f(\theta)\,d\theta$$
$$- \left[\int_{\underline{\theta}}^{\theta} q(x)\,dx F(\theta) \,|_{\theta=\underline{\theta}}^{\theta=\overline{\theta}} - \int_{\underline{\theta}}^{\overline{\theta}} F(\theta)\, q(\theta)\,d\theta \right]$$

Since $F\left(\overline{\theta}\right) = 1$, the objective function becomes:

$$\int_{\underline{\theta}}^{\overline{\theta}} (\theta q(\theta) - C(\theta))\, f(\theta)\,d\theta$$
$$- \int_{\underline{\theta}}^{\overline{\theta}} q(x)\,dx + \int_{\underline{\theta}}^{\overline{\theta}} F(\theta)\, q(\theta)\,d\theta$$

Taking $f(\theta)\,d\theta$ out of each term yields:

$$\int_{\underline{\theta}}^{\overline{\theta}} \left[\theta q(\theta) - C(q(\theta)) - q(\theta)\frac{1-F(\theta)}{f(\theta)}\right] f(\theta)\,d\theta$$

$$= \int_{\underline{\theta}}^{\overline{\theta}} \left[\underbrace{\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right)}_{virtual\ type} q(\theta) - C(q(\theta))\right] f(\theta)\,d\theta$$

This objective function we can maximize pointwise. The FOCs are:

$$\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) = C'(q(\theta))$$

Since C is convex, $q(\theta)$ is increasing if and only if $\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right)$ is increasing, which is the case if the hazard rate $\frac{f(\theta)}{1-F(\theta)}$ is increasing. The hazard rate is increasing for many probability distributions: normal, uniform, Pareto, logistic, exponential, etc.

Properties of the solution:

- The high type gets an efficient allocation of the commodity.
- All other types get subefficient amounts of the commodity.
- All types, except for the lowest type, get positive information rents, which increases with the type.
- The lowest type gets a zero surplus.
- In general little can be said about the shape of $T(q)$ schedule.

3.4 Ironing and Bunching

Since C is convex, $q(\theta)$ is increasing if and only if $\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right)$ is increasing, which is the case if the hazard rate $\frac{f(\theta)}{1-F(\theta)}$ is increasing. The hazard rate is increasing for many probability distributions: normal, uniform, Pareto,

logistic, exponential, etc. When the condition that q(θ) is increasing is violated, it becomes necessary to "iron" the schedule $q(\theta)$ by assigning the same contract to a set of individuals also known as bunching. Sometimes bunching provides the basis of empirical tests of the theory. For a rigorous treatment of the subjects, please see pp. 88-93 in Bolton and Dewatripont (2005).

3.5 Optimal Deterministic Mechanism

In the analysis above, we have considered only deterministic contracts based on one's type. However, under the assumptions of the model stated above, the optimal contract is deterministic. is strictly concave because $C(q(\theta))$ is strictly convex

Proposition: *Under the assumptions of the model, the solution of the Principal's problem is deterministic and unique.*

Proof. Suppose in contradiction that there is a stochastic solution $(Q(\theta), T(\theta))$ defined by a probability distribution g over $(q(\theta), t(\theta))$and let $(E(Q(\theta)), E(T(\theta)))$ be the deterministic contract obtained by taking expectations of the random terms in the optimal solution.The objective function

$$\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) q(\theta) - C(q(\theta))$$

is strictly concave because C($q(\theta)$) is strictly convex. By Jensen's inequality,

$$\int_{\underline{\theta}}^{\overline{\theta}} E_g \left[\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) q(\theta) - C(q(\theta))\right] f(\theta)\, d\theta$$

$$< \int_{\underline{\theta}}^{\overline{\theta}} \left[\left(\theta - \frac{1-F(\theta)}{f(\theta)}\right) E(Q(\theta)) - C(E(Q(\theta)))\right] f(\theta)\, d\theta$$

which is a contradiction with the assumption that $(Q(\theta), T(\theta))$ was optimal. ■

4 Perfect Competition in Contracts

Observe that profits are positive for all types in both the two-type model and the model with a continuum of agents. For simplicity, here we consider

only the two-type model. Recall that the Principal extracts all the surplus from the low type:

$$\pi_1 = \theta_1 q_1 - C(q_1)$$

and the profits on the high type are:

$$\pi_2 = \theta_2 (q_2 - q_1) + \theta_1 q_1 - C(q_2)$$

Therefore:

$$\begin{aligned} & \pi_1 - \pi_2 \\ = \; & \theta_2 (q_2 - q_1) - (C(q_2) - C(q_1)) \\ = \; & \int_{q_1}^{q_2} (\theta_2 - C'(q))\, dq > 0 \end{aligned}$$

Since $\theta_2 \geq C'(q)$ for all $q \in [0, \theta_2]$. Thus, the profits on the high type are greater than the profits on the low type. Now suppose that new Principals may enter freely in the market (i.e. there are no entry costs). The new Principals will be willing to enter as long as they make non-zero profits. For example, the new entrant may offer just slightly better conditions to θ_2 than the incumbent and, as a result, capture the high end of the market. The incumbent is, however, also likely to respond to this threat. The two will Bertrand compete driving their profits on the high type to 0. Thus, we observe that allowing for free entry of new Principals reduces the profits of the incumbent. In what follows, we investigate the magnitude of the profit reduction.

For the purposes of answering this question, we have to use an equilibrium concept. We follow the Rothschild -Stiglitz equilibrium defined below.

Definition. Let a profitable entrant be a contract that makes positive profits if all other existing contracts remain unchanged. Then, the Rothschild -Stiglitz equilibrium is defined as the set of contracts, such as there is no profitable entrant and no contract leads to losses.

That is, the Rothschild -Stiglitz equilibrium is a Nash equilibrium of a game played by the Principals, where the strategies of the players are contracts.

Proposition: The Irrelevance Result *With free entry of Principals, the set of Rothschild -Stiglitz equilibria under complete and asymmetric information are the same.*

Proof. First, consider the case of complete information. The Principals know the type of the agent and Bertrand compete, which drives the transfer

down to the cost of production. Thus, the Principals offer a menu of contracts $(q^*(\theta), t^*(\theta))$, such that $(q^*(\theta), t^*(\theta))$ solves the profit maximization problem of the firm and that has the following properties:

- $t^*(\theta) = C(q^*(\theta))$;
- $q^*(\theta)$ is the solution of $\theta = C'(q^*(\theta))$.

By the convexity of the cost function, the equilibrium is unique.

Next, consider the case of asymmetric information. We will prove that menu of contracts $(q^*(\theta), t^*(\theta))$ is the unique Rothschild -Stiglitz equilibrium. Observe that $(q^*(\theta), t^*(\theta))$ allows type θ to extract all the surplus from the relation with the Principal. Consequently, θ will choose from the menu of contracts, the contract corresponding to her type. This applies for all types. Consequently, the menu of contracts $(q^*(\theta), t^*(\theta))$ is incentive compatible and, obviously, individually rational. Furthermore, since the profits of the incumbent principals are again zero, there are no entries. Thus, the menu of contracts $(q^*(\theta), t^*(\theta))$ constitutes a Rothschild -Stiglitz equilibrium. It is unique from the convexity of the costs.

5 Extensions

The model that we considered above extends with some modifications to the analysis of a number of environments:

- Credit rationing
- Optimal taxation
- Regulation
- Insurance
- Reforming the public sector
- Labor contracts, etc.

In this course, we will focus on the first five cases. If interested in the last topic and a general overview of the literature, please read Chapter 13 B of Mas-Colell, Whinston, and Green.