Ecole Polytechnique
ECO 560 Economics of Contracts

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Lecture 1: Introduction to Contract Theory and Mechanism Design

1 Introduction

Modern economics was born in the second half of the 1940s. In contrast to previous work in philosophy, sociology, and political economy, it committed itself to a study based on mathematical modelling and the scientific methods of the natural sciences. This approach has some advantages, in particular logical consistency of the study and falsifiability. At the same time, it has limitations: among others, the tension between reality and assumptions. These are particularly evident in the field of economics.

In the early 1950s, Arrow, Debreu, and McKenzy formalized prior work by Walras and proved the existence of a General Equilibrium. Their work provided the foundations of modern macroeconomics and appeared to provide consistent and quite general foundations for the study of economic phenomena around some basic notions first introduced by Smith, Ricardo, and Walras, and revived by Samuelson in the mid 1940s. In the 1960s and the 1970s, however, economists became more and more aware that the general equilibrium framework omits some features of the environment that have profound effects on economic outcomes. One of the phenomena that the General Equilibrium theory could not account for is the fact that considerable part of the economic activity takes place not at the market place but between

parties whose interactions are regulated by contracts. Contract theory has attempted to explain at least some of these phenomena by building models that relax two crucial assumptions of the General Equilibrium theory: (1) that agents do not act strategically to influence prices and others' behavior; and (2) that all agents have the same information about the economy. In particular, contract theory allows for both strategic interaction between agents and differences in the information that each agent has about the economy. The complexity of this more general setting forced, at least initially, contract theorists to restrict themselves mostly to models that share the following features:

- partial equilibrium models: focus on the market for one good, in isolation from the rest of the economy;
- small number of agents, usually two;
- the models sum up the constraints imposed by the prevailing institutional setting through a contract, explicit (enforced by a third party) or implicit (sustained by a Nash equilibrium, involving the agreeing parties).

Also, for the purposes of the course, we will define information to be either who the agents are (hidden information) ot what the agents do (hidden action). Most models in contract theory, and definitely all models that we will review in the course, rely on a very simple framework. Formally, this framework is known as the Principal-Agent Framework/Paradigm/Model. It has the following characteristics:

- There are two players: an informed player, whose information is relevant to the outcome of her interaction with an uninformed player.
- The two players engage in bargaining of a very simple type: all bargaining power is allocated to one of the players, who offers a "take it or leave it" contract.
- That is, the players engage in a Stackelberg game: the leader, called the Principal, proposes the contract and the follower, called the Agent, accepts or rejects it.

Within the Principal-Agent framework, we will study three types of models:

- Screening models. Models with hidden information in which the uninformed player moves first to offers a contract to the uninformed player.

- Signalling models. Models with hidden information in which the informed player moves first to take an action (usually send a signal) and then the uninformed player responds.

- Moral hazard models. Models with hidden action in which the uninformed player acts first (usually offers a contract) and the informed player chooses an action.

Example screening

Setting: Consider the market for detergent in retail stores. There are different customer types: single, couples, family with children, etc. The characteristics of demand vary with the types. The seller does not observe the types. Besides, discrimination is not legal...

Q: How can the seller vary the price and quantity sold to elicit the hidden information from the customer in a profit-maximizing way?

A: Uninformed party, the seller, moves first and offers a schedule of quantities and prices. The informed party acts second and decides to buys or not to buy.

Example: signalling

Setting: Consider the market for used cars. The quality of the cars vary and is known to the owners, but not to the potential buyers.

Q: How can an owner of a high quality car reveal its type to the buyers?

A: The informed party, the seller, moves first and sends a costly or costless signal to the buyers.

Example: moral hazard

Setting: Consider an employment relation in which the effort exerted by the employee is either not observable or not verifiable by a third party.

Q: What compensation schedule can the firm offer to induce its employees to work?

A: The uninformed party moves first and offers a compensation schedule. Then, the informed party chooses action.

2 Mechanism Design: Overview

In many setting where a collective decision needs to be made, the preferences of the agent are not publicly observed. In such settings, the individual must be relied to reveal this information. The problem of eliciting this information from the individual is known as mechanism design problem. Loosely speaking, mechanism design is about the design of the (Bayesian) game that the agent play in order to make a collective action.

Example 1: A community must decide to build or not to build a bridge. The personal valuation of the bridge, however, is something that each agent only knows. Of course, if all agents value the bridge so much that the benefits outweigh the costs, it should be build. In many cases, agents may not be willing to reveal their true valuation. In particular, if high declared valuation implies also high contribution to the costs. Ideally, one would like to enjoy the services of the bridge without incurring the costs. More abstractly, the agents play a game whose payoffs depend on the collective decision. In this context, the problem is to determine the rules of the game, so that the collective decision is to build the bridge when the true benefits are greater than the cost and not otherwise. Another important problem is the cost of building the bridge under the mechanism.

2.1 Introduction and Definitions

Let the players be $i = 1, ..., I$ and Y the set from which collective choice is made. Each individual is characterized by a type θ_i and a set of possible types Θ_i. $\theta_i \in \Theta_i$ and only i knows it. However, i can declare to be any type in Θ_i. The agent utility is a function of the allocation $y \in Y$ and one's type: $u_i(y, \theta_i)$. The possible types from the product space $\Theta_1 \times ... \times \Theta_I$ and a prior distribution over the types ϕ are common knowledge. Again, θ_i is private information.

The collective decision is modeled as a social choice function that takes as its argument the types of the players θ and maps them to an allocation y. Formally,

Def 1: A social choice function is $f : \Theta_1 \times ... \times \Theta_I \rightarrow Y$, i.e. $f(\theta) \in Y$.

The collective decision is ex-post efficient if the resulting allocation is efficient, i.e. there is no other allocation that gives more utility to some players without decreasing the utility of the others. Formally,

Def 2: A social choice function f is ex-post efficient if for no θ, there exists y, such that $u_i(f(\theta), \theta_i) \leq u_i(y, \theta_i)$ with at least one strict inequality.

Let us state the bridge example from above formally.

Example 1, contd. The community has to decide whether to build a bridge (outcome 1) or not (outcome 0): $k = \{0, 1\}$.. In case, they decide to build the bridge, each individual has to make a transfer t_i. The transfer can be negative or positive: when negative, the individual pays and when positive she receives money. Thus, an element of the allocation space Y, y is $(k(\theta), t_1(\theta), ..., t_I(\theta))$. Note that both the transfers and the outcomes are functions of the types. $\theta_i \in \Theta_i = [0, +\infty)$ Let c is the cost of the projects and define the individual utility as

$$u_i(y, \theta_i) = \theta_i k - t_i$$

Part A: The social choice function $f(\theta)$ is ex-post efficient if, for all θ,

$$k(\theta) = 1 \text{ if } \sum_i \theta_i \geq c \text{ and } 0 \text{ otherwise.}$$

$$\sum_i t_i(\theta) = ck(\theta)$$

That is, the bridge is built if the total benefit is greater than the cost. Furthermore, it must be the case, that the when the bridge is built, the total transfers from the population equal the cost of the bridge and when it is not, the total transfers add up to 0. Note that in the latter case, this does not imply that all transfers are 0.

Part B. Next, we go to the heart of the problem: will the agents report truthfully their types for any mechanism, when the types are private information? Suppose that we would like to implement a social choice function that is ex-post efficient in the sense above and that is also egalitarian in the sense that

$$t_i(\theta) = \frac{ck(\theta)}{I}.$$

Further suppose that all types are known, except for the type of agent 1. To make the example interesting, we also maintain that:

$$c > \sum_{i=2}^{I} \theta_i > \frac{(I-1)c}{I}$$

Then $k(\theta) = 1$ is socially efficient if

$$\theta_1 \geq c - \sum_{i=2}^{I} \theta_i$$

Suppose that

$$\theta_1 = c - \sum_{i=2}^{I} \theta_i + \varepsilon$$

where ε is very small. That is, the bridge should be built but the total social gain is barely greater than the cost. Would agent 1 tell the truth? When telling the truth, she receives:

$$c - \sum_{i=2}^{I} \theta_i + \varepsilon - \frac{c}{I}$$

which for sufficiently small ε is negative. Thus, agent 1 would declare a type, such that the bridge is not built.

Next, we define mechanism. Mechanism can be regarded as a system of rules that govern the procedure of a collective decision. Formally, the definition is below.

Def 3: A mechanism Γ is $\{S_1, ..., S_I, g\}$, where S_i is a strategy space and $g : S_1 \times ... \times S_I \rightarrow Y$

As you see, the mechanism stipulates what agents can do, i.e. defines their strategy spaces, and how their choice of actions determine the outcomes of the collective decision, i.e . it also defines the outcome function g. Simply put, the mechanism stipulates what agents can do and how their actions will be rewarded. In this sense, it is very close to defining a game. In combination with the collection of possible types for each player, the related common prior, and the utility functions, a mechanism Γ induces a Bayesian game

$$\{I, \{S_i\}, \{u_i(g(s), \theta_i), \Theta_1, ..., \Theta_I, \phi\}\}$$

Finally, we connect mechanism to the notion of social choice function below.

Definition 4: The mechanism Γ implements social choice function f if there is an equilibrium strategy profile $(s_1^*(.), ..., s_I^*(.))$ in the induced game, such that

$$g(s_1^*(\theta_1), ..., s_I^*(\theta_I)) = f(\theta_1, ..., \theta_I)$$

for all $(\theta_1, ..., \theta_I) \in \Theta_1 \times ... \times \Theta_I$.

Note that we did not specify the solution concept that will be used in the game. This is because different concepts are often used.

The key problem in mechanism design is the identification of all social choice functions that can be implemented through some mechanism. The space of possible mechanism is potentially huge, so the task is in principle difficult. However, there is a very nice result known as the revelation principle that states that in some settings we can restrict our attention to a very simple type of mechanisms, which we define below.

A very special family of mechanisms is defined below. These will be very important in the following subsection and the rest of the course.

Definition 5: A direct revelation mechanism Γ is such that $\mathrm{S}_i = \Theta_i$ and $\mathrm{g}(\theta) = f(\theta)$ for all $\theta \in \Theta_1 \times ... \times \Theta_I$.

In words, a direct revelation mechanism is a mechanism in which the economic agents are allowed to make statements about their types. A truthful direct mechanism is a mechanism that implements a social choice function, so that declaring the true types of the economic agents is an equilibrium of the induced game.

2.2 Mechanism Design: Application to Screening Models

The Principal is the designer or Center. $I = 1$. Given a mechanism $\{S, g\}$, the Agent chooses the message he sends so as to maximize his utility.

$$s^*(\theta) \in \arg\max_{s \in S} u(g(s), \theta)$$

and obtains allocation $g^*(\theta) = g(s^*(\theta))$. The problem that the Principal faces is to determine which allocations of the surplus generated by the relation with the Agent are implementable by a mechanism. The space of mechanisms is large, so this may appear to be a very difficult task. The Revelation Principle described below, however, shows how in the context that we study the mechanism design problem can be simplified dramatically.

Revelation Principle: If the allocation $f^*(\theta)$ can be implemented through some mechanism, then it can be implemented through a direct truthful mechanism.

Proof: Let $\Gamma = \{g, S\}$ be a mechanism that implements f. Let s^* be an equilibrium strategy that implements $f(\theta)$, where θ is the truthful profile

of types: $g(s^*(\theta)) = f(\theta)$. Consider the direct mechanism $\{f(), \Theta\}$ and suppose in contradiction that it does not implement the allocation $f^*(\theta)$. That is, the agent would prefer to announce some θ' rather than his true type θ:

$$u\left(f\left(\theta^{true}\right), \theta\right) < u\left(f\left(\theta'\right), \theta\right)$$

However, from the condition $g(s^*(\theta)) = f(\theta)$, this would imply

$$u\left(g\left(s^*\left(\theta^{true}\right)\right), \theta\right) < u\left(g\left(s^*\left(\theta'\right)\right), \theta\right)$$

Consequently, we arrived at a contradiction.■

Thus, the Principal can limit his search to direct mechanisms i.e. to mechanisms that require the Agent just to make a statement about her type, usually in the sense of choosing a particular contract from a menu of contracts offered by the Principal.

3 Screening or Adverse Selection

In this lecture we cover a basic screening model with two types. We start with a model of screening with two types because this allows us to appreciate the features of the model in a friendly, noncomplicated environment. Then, we cover the general model of screening with a continuum of types. Both models have been based on Mussa and Rosen (1978). I consider only deterministic contracts here: the proof that the optimal menu of contracts is deterministic under some general conditions is left for Problem Set 1.

4 Screening: Model with Two Types

4.1 Setting

Consider a seller who faces two types of customers: with probability π he faces customer of type θ_1 and $(1-\pi)$ customer of type θ_2. Let q be the quality purchased and t be the transfer made from the buyer to the seller. The buyer's utility is $U(\theta_i, (q,t)) = \theta_i q - t$.The seller's cost is C($q$), where C($q$) is convex, twice differentiable, C'(0) = 0 and C'(∞) = ∞ The seller's profits from serving a customer are $t - C(q)$.

To find the optimal pair of contracts, we solve

$$\max_{t_1,q_1,t_2,q_2} \{\pi (t_1 - C(q_1)) + (1-\pi)(t_2 - C(q_2))\}$$

s.t.

$$\theta_1 q_1 - t_1 \geq \theta_1 q_2 - t_2 \quad (IC1)$$
$$\theta_2 q_2 - t_2 \geq \theta_2 q_1 - t_1 \quad (IC2)$$
$$\theta_1 q_1 - t_1 \geq \underline{u} \quad (IR1)$$
$$\theta_2 q_2 - t_2 \geq \underline{u} \quad (IR2)$$

Normalizing the reservation utility to 0, we have problem **(P)**:

$$\max_{t_1,q_1,t_2,q_2} \{\pi (t_1 - C(q_1)) + (1-\pi)(t_2 - C(q_2))\}$$

s.t.

$$\theta_1 q_1 - t_1 \geq \theta_1 q_2 - t_2 \quad (IC1)$$
$$\theta_2 q_2 - t_2 \geq \theta_2 q_1 - t_1 \quad (IC2)$$
$$\theta_1 q_1 - t_1 \geq 0 \quad (IR1)$$
$$\theta_2 q_2 - t_2 \geq 0 \quad (IR2)$$

4.2 Benchmark Case: Types are known

When the Principal knows the type of the Agent, he can extract all the surplus from the relation:

$$t_1 = \theta_1 q_1$$
$$t_2 = \theta_2 q_2$$

Since the Principal can perfectly identify the customer, the Principal's problem for customer i becomes:

$$\max_{q_i} \theta_i q_i - C(q_i)$$

which of course implies that the optimal q_i^* is the solution to

$$\theta_i - C'(q_i) = 0.$$

However, if the Principal does not know the type of the buyer, then the contracts above would not help the Principal find out what is the type of the customer. Note that from above:

$$0 = \theta_1 q_1^* - t_1 < \theta_2 q_1^* - t_1$$

$$0 = \theta_2 q_2^* - t_2 < \theta_2 q_1^* - t_1$$

so type 2 would have an incentive to present herself as type 1.

4.3 Solving the Model 1

Proposition 1: At the optimum for program **(P)**:

1. IR 1 binds, so $t_1 = \theta_1 q_1$.
2. IC 2 binds, so $t_2 - t_1 = \theta_2(q_2 - q_1)$.
3. $q_2 \geq q_1$
4. q_2 is equal to the efficient quantity (no distortion on top).
5. IC 1 and IR 2 are redundant.

   Proof: The problem becomes, after the normalization of $\underline{u} = 0$.

$$\max_{t_1, q_1, t_2, q_2} \{\pi (t_1 - C(q_1)) + (1 - \pi)(t_2 - C(q_2))\}$$

s.t.

$$\theta_1 q_1 - t_1 \geq \theta_1 q_2 - t_2 \quad (IC1)$$

$$\theta_2 q_2 - t_2 \geq \theta_2 q_1 - t_1 \quad (IC2)$$

$$\theta_1 q_1 - t_1 \geq 0 \quad (IR1)$$

$$\theta_2 q_2 - t_2 \geq 0 \quad (IR2)$$

The revalation principle stipulates that if an allocation (q, t) can be implemented through a menu of contracts, then it can also be implemented through a menu of contracts that includes only one offer for each type. Moreover, the solution is such that each type finds it in her interest to choose the offer associated with her true type. Show that if the seller is maximizing profits when types are unknown, it must be the case that he offers:

1. **Proof:** From IC 2:

$$\theta_2 q_2 - t_2 \geq \theta_2 q_1 - t_1$$

Since $\theta_1 < \theta_2$

$$\theta_2 q_1 - t_1 \geq \theta_1 q_1 - t_1$$

Suppose that IR 1 does not bind

$$\theta_1 q_1 - t_1 > 0$$

From above this implies that

$$\theta_2 q_2 - t_2 > 0$$

Consequently, we can increase t_1 and t_2 by the same amount, preserve the choice of the type, and increase the profits of the Principal. Thus, the original menu of contracts is not optimal, which contradicts the assumptions. Thus, IR 1 binds and $t_1 = \theta_1 q_1$.

2. **Proof:** Suppose in contradiction that IC 2 does not bind, i.e.

$$\theta_2 q_2 - t_2 > \theta_2 q_1 - t_1$$

By property 1 and since $\theta_1 < \theta_2$:

$$\theta_2 q_1 - t_1 \geq \theta_1 q_1 - t_1 = 0$$

Again, we can increase t_2 by a small amount to increase profits, while preserving incentive compatibility and individual rationality. Again, this implies a contradiction with the assumption that the proposed menu of contracts is optimal. Thus, IC 2 binds so

$$t_2 - t_1 = \theta_2(q_2 - q_1).$$

By property 1 then, we can eliminate

$$t_2 = \theta_2(q_2 - q_1) + \theta_1 q_1$$

3. **Proof:** Draw the indifference curves for types θ_1 and θ_2 that go through (q_1, t_1) in a $q - t$ coordinate system and determine the preferred sets for each type. The previous part indicates that (q_1, t_1) must belong to the indifference curve that goes through the origin giving the IR

level of utility 0. Further, the optimal bundle for θ_2 must belong to the indifference curve for the θ_2 that goes through (q_1, t_1). Since both IC_2 and IR_1 need to be satisfied at the optimum, the optimal offer for the high type must belong to her indifference curve that goes through (q_1, t_1). We will show that $q_2 > q_1$. Note that under t_1 and t_2 derived in 1 and 2, such that IR_1 and IC_2 are satisfied, the first best quantities $\left(q_1^{FB}, q_2^{FB}\right)$ can be achieved even when types are not observed. Note that $\left(q_1^{FB}, t_1^{FB}\right)$ satisfied IR_1. To ensure that incentive compatibility is also satisfied, we need to draw an indifference curve through $\left(q_1^{FB}, t_1^{FB}\right)$. This indifference curve intersects the vertical line $q = q_2^{FB}$. The transfer is

$$t_2 = \theta_2(q_2^{FB} - q_1^{FB}) + \theta_1 q_1^{FB}$$

The question is whether this schedule is still optimal. $\left(q_1^{FB}, q_2^{FB}\right)$ are the maximizers of the profit function when types are known:

$$\pi\theta_1 q_1 + (1-\pi)\,\theta_2 q_2 - \pi C\,(q_1) - (1-\pi)\,C\,(q_2)$$

where π is the probability of the low type. However, the objective function of the Principal when types are **not known** is:

$$\pi\theta_1 q_1 + (1-\pi)\,\theta_2 q_2 - (1-\pi)\,(\theta_2 - \theta_1)\,q_1 - \pi C\,(q_1) - (1-\pi)\,C\,(q_2)$$

That is, there is the term $-(1-\pi)\,(\theta_2 - \theta_1)\,q_1$ which implies that the optimal level for the low type when types are not knwon is less than q_1^{FB}. Thus, we have that the optimal (q_2, t_2) is such that $q_2 > q_1$. By the single-crossing property, it belongs to the set of bundles (q, t) which are weakly prefered by the high type but are dominated by (q_1, t_1) from the perspective of θ_1.

4. **Proof:** The result follows from the first-order conditions for optimality when types are not known.

5. **Proof:** From 3, IC_1 holds for the optimal contract. IR_2 also holds since

$$\theta_1 q_1 - t_1 < \theta_2 q_1 - t_1 = \theta_2 q_2 - t_2$$

'Purely Algebraic' solution

Proof: From IC 2:

$$\theta_2 q_2 - t_2 \geq \theta_2 q_1 - t_1$$

Since $\theta_1 < \theta_2$

$$\theta_2 q_1 - t_1 \geq \theta_1 q_1 - t_1$$

Suppose that IR 1 does not bind

$$\theta_1 q_1 - t_1 > 0$$

From above this implies that

$$\theta_2 q_2 - t_2 > 0$$

Consequently, we can increase t_1 and t_2 by the same amount, preserve the choice of the type, and increase the profits of the Principal. Thus, the original menu of contracts is not optimal, which contradicts the assumptions. Thus, IR 1 binds.

Suppose in contradiction that IC 2 does not bind, i.e.

$$\theta_2 q_2 - t_2 > \theta_2 q_1 - t_1$$

By property 1 and since $\theta_1 < \theta_2$:

$$\theta_2 q_1 - t_1 \geq \theta_1 q_1 - t_1 = 0$$

Again, we can increase t_2 by a small amount to increase profits, while preserving incentive compatibility and individual rationality. Again, this implies a contradiction with the assumption that the proposed menu of contracts is optimal. Thus, IC 2 binds so

$$t_2 - t_1 = \theta_2(q_2 - q_1).$$

By property 1 then, we can eliminate t_1

$$t_2 = \theta_2(q_2 - q_1) + \theta_1 q_1$$

Let us add IC 1 and IC 2. The transfers cancel out and we are left with:

$$\theta_2 \left(q_2 - q_1\right) \geq \theta_1 \left(q_2 - q_1\right)$$

$$\left(\theta_2 - \theta_1\right)\left(q_2 - q_1\right) \geq 0$$

Since $\theta_2 > \theta_1$, $q_2 - q_1 \geq 0$.

Since both IC_2 and IR_1 need to be satisfied at the optimum:

$$t_1 = \theta_1 q_1$$

$$t_2 = \theta_2(q_2 - q_1) + \theta_1 q_1$$

Note that under t_1 and t_2 the first best quantities $\left(q_1^{FB}, q_2^{FB}\right)$ can be achieved even when types are not observed: the question is whether these quantities are still optimal. In particular, IR_1 is satisfied for the first best $\left(q_1^{FB}, \theta_1 q_1^{FB}\right)$. Next, observe that

$$(q_2^{FB}, \theta_2(q_2^{FB} - q_1^{FB}) + \theta_1 q_1^{FB})$$

satisfies IC_2 with an equality. We will show that the optimal q_1 when types are unknown is less than q_1^{FB}. $\left(q_1^{FB}, q_2^{FB}\right)$ are the maximizers of the following profit function when types are known:

$$\pi\theta_1 q_1 + (1-\pi)\,\theta_2 q_2 - \pi C\,(q_1) - (1-\pi)\,C\,(q_2)$$

Using IR_1 and IC_2, we can simplify the objective function of the Principal when types are not known to:

$$\pi\theta_1 q_1 + (1-\pi)\,\theta_2 q_2 - (1-\pi)\,(\theta_2 - \theta_1)\,q_1 - \pi C\,(q_1) - (1-\pi)\,C\,(q_2)$$

Consider the unconstrained optimization of this function with respect to (q_1, q_2). Note that, we have the term $-(1-\pi)\,(\theta_2 - \theta_1)\,q_1$ which implies that the solution for q_1 is less than q_1^{FB}. At the same time, the solution for q_2 is equal to q_2^{FB}. We will show that the solution of this problem, (q_1^*, q_2^*), satisfies IC_1 and IR_2. Rewriting IC 1

$$\theta_1(q_2 - q_1) \leq t_2 - t_1$$

From property 2, we have that

$$t_2 - t_1 = \theta_2(q_2 - q_1)$$

From above $(q_2^* - q_1^*) \geq 0$. Thus, IC 1 is satisfied at (q_1^*, q_2^*).

Furthermore, from IC 2:

$$\theta_2 q_2 - t_2 = \theta_2 q_1 - t_1$$

Since $\theta_1 < \theta_2$

$$\theta_2 q_1 - t_1 > \theta_1 q_1 - t_1 = 0$$

Thus, if IR 1 holds, i.e.

$$\theta_1 q_1^* - t_1 = 0$$

implies that

$$\theta_2 q_2^* - t_2 > 0.$$

Suppose in contradiction that the optimal q_2 is different from the efficient quantity that we found in the benchmark case. From the properties of the cost function, this implies that

$$C'(q_2) < \theta_2$$

Now consider an alternative menu of contracts, where $\varepsilon > 0$ is sufficiently small: $(q_1, t_1), (q_2' = q_2 + \varepsilon, t_2' = t_2 + \varepsilon\theta_2)$. Note that

$$\theta_2 q_2' - t_2' = \theta_2 q_2 - t_2$$

so the IC 2 and IR 2 is satisfied as before. Since

$$\theta_1 q_2' - t_2' = \theta_1 q_2 - t_2 - \varepsilon(\theta_2 - \theta_1)$$

so that IC 1 is also satisfied. However,

$$t_2' - C(q_2') \simeq t_2 - C(q_2) + \varepsilon(\theta_2 - C'(q_2))$$

Consequently, the profits under the new menu of contracts are higher than the original, a contradiction. ■

4.4 Solving the Model 2

By Proposition 1, we can modify the problem as follows:

$$\max_{t_1, q_1, t_2, q_2} \{\pi(t_1 - C(q_1)) + (1 - \pi)(t_2 - C(q_2))\}$$

s.t.

$$\theta_2 q_2 - t_2 = \theta_2 q_1 - t_1 \quad (IC2)$$

$$\theta_1 q_1 - t_1 = \underline{u} \ (IR1)$$

where

$$t_1 = \theta_1 q_1$$

$$t_2 = \theta_2(q_2 - q_1) + \theta_1 q_1$$

Substituting the equality constraints into the objective function, we have to solve the following problem of unconstrained optimization:

$$\max_{q_1,q_2} \{\pi (\theta_1 q_1 - C(q_1)) + (1-\pi)(\theta_2(q_2 - q_1) + \theta_1 q_1 - C(q_2))\}$$

Reorganizing terms, we have:

$$\max_{q_1,q_2} \{\pi (\theta_1 q_1 - C(q_1)) + (1-\pi)(\theta_2 q_2 - C(q_2) - (\theta_2 - \theta_1) q_1)\}$$

Thus, the first term in the objective function is equal to the full surplus from the market exchange with type 1, while the second term is equal to less than the full surplus of the market exchange with type 2. The term $(\theta_2 - \theta_1) q_1$ is the information rent that type 2 obtains in return for revealing her type.

Solution. From above, we already know that the optimal q_2 is the efficient level. This can be seen again from the FOC for q_2:

$$C'(q_2) = \theta_2$$

The FOC for q_1 is:

$$\theta_1 - \frac{1-\pi}{\pi}(\theta_2 - \theta_1) = C'(q_1)$$

Since

$$\theta_1 - \frac{1-\pi}{\pi}(\theta_2 - \theta_1) < \theta_1$$

so type 1 receives less than the efficient level of commodity 1.

4.5 Properties of the solution:

- The high type gets an efficient allocation of the commodity.
- All other types get subefficient amounts of the commodity.
- All types, except for the lowest type, get positive information rents, which increases with the type.
- The lowest type gets a zero surplus.
- In general little can be said about the shape of $T(q)$ schedule.